# BASIS FOR TRANSIENT METHODS OF DETERMINING THERMOPHYSICAL CHARACTERISTICS

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A two-dimensional solution is derived for a limited cylinder subject to boundary conditions of the first kind; it is convenient for calculations involving small Fourier numbers. The solution is used to provide a basis for transient methods of determining the thermophysical characteristics of nonmetallic materials using models of semiinfinite bodies.

The model of a semiinfinite body is often used in transient methods of determining the thermophysical properties of materials [2-4]. Real samples possess finite dimensions in every direction, however, and the condition of "semiinifinite" may not always be satisfied, or may simply be satisfied up to a certain instant of time, depending on the relationship between the geometrical dimensions of the sample, the disposition of the sensors, the temperatures of the latter, the thermophysical properties of the sample itself, and also the conditions of heat transfer between the test sample and the surrounding medium.

This paper is devoted to making a theoretical assessment of the extent to which the semiinfinite condition is satisfied for a finite cylinder. A detailed table will be given for the relative excess temperature, expressed as a function of the local Fourier number; this table may be used for calculating thermal diffusivity in an experimental method based on the principles outlined by Lykov [1] for a semiinfinite body with boundary conditions of the first kind. The graphical material presented (as well as the table) were obtained using a Minsk-22 computer.

Let us suppose that we have a finite cylinder $0 < x < l$ and $0 \le r \le R$; at the initial instant of time the temperature is constant and equal to that of the surrounding medium $T_0$. At a slightly later instant $\tau > 0$ the surface $x = 0$ takes a temperature $T_C \ne T_0$, which is then held constant during the whole heat-transfer process, while the surfaces $x = l$ and $r = R$ remain at the initial temperature.

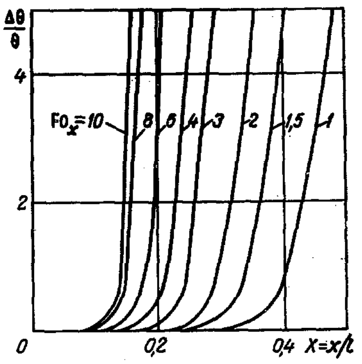


Fig. 1

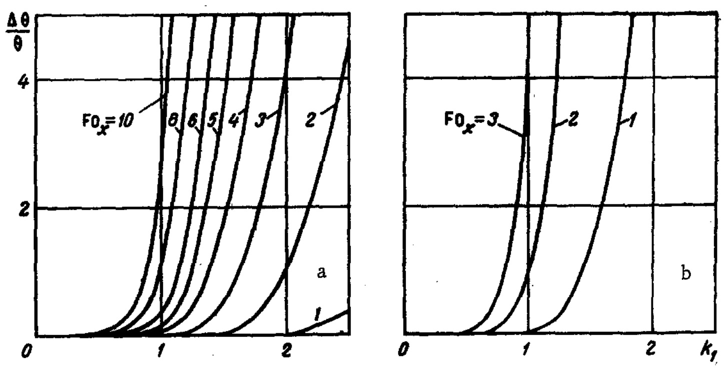


Fig. 2

Fig. 1. Dependence of the relative error $\Delta\theta/\theta_\infty$, % on the parameter X for an infinite plate and various values of the local Fourier numbers $Fo_X$.

Fig. 2. Dependence of the relative error $\Delta\theta/\theta_\infty$, % on the parameter $k_1$ for X = 0.1 (a) and X = 0.2 (b) on the axis of an infinite cylinder (r = 0) and various values of the local Fourier number $Fo_X$.

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TABLE 1. Values of the Function erf = $1/2\sqrt{Fo_x} = \theta_\infty$ in Relation to the Fourier Number $Fo_x$ for a Semiinfinite Body

| $Fo_x$ | $\theta_\infty$ | $Fo_x$ | $\theta_\infty$ | $Fo_x$ | $\theta_\infty$ |
|---|---|---|---|---|---|
| 0,01 | 1,0000 | 0,57 | 0,6510 | 1,65 | 0,4180 |
| 0,02 | 1,0000 | 0,58 | 0,6468 | 1,70 | 0,4124 |
| 0,03 | 1,0000 | 0,59 | 0,6427 | 1,75 | 0,4070 |
| 0,04 | 0,9996 | 0,60 | 0,6387 | 1,80 | 0,4018 |
| 0,05 | 0,9984 | 0,61 | 0,6347 | 1,85 | 0,3968 |
| 0,06 | 0,9961 | 0,62 | 0,6308 | 1,90 | 0,3920 |
| 0,07 | 0,9925 | 0,63 | 0,6270 | 1,95 | 0,3874 |
| 0,08 | 0,9876 | 0,64 | 0,6232 | 2,00 | 0,3829 |
| 0,09 | 0,9816 | 0,65 | 0,6195 | 2,05 | 0,3786 |
| 0,10 | 0,9747 | 0,66 | 0,6159 | 2,10 | 0,3744 |
| 0,11 | 0,9670 | 0,67 | 0,6123 | 2,15 | 0,3704 |
| 0,12 | 0,9588 | 0,68 | 0,6088 | 2,20 | 0,3664 |
| 0,13 | 0,9501 | 0,69 | 0,6054 | 2,25 | 0,3626 |
| 0,14 | 0,9412 | 0,70 | 0,6020 | 2,30 | 0,3590 |
| 0,15 | 0,9321 | 0,71 | 0,5986 | 2,35 | 0,3554 |
| 0,16 | 0,9229 | 0,72 | 0,5953 | 2,40 | 0,3519 |
| 0,17 | 0,9137 | 0,73 | 0,5921 | 2,45 | 0,3486 |
| 0,18 | 0,9044 | 0,74 | 0,5889 | 2,50 | 0,3453 |
| 0,19 | 0,8952 | 0,75 | 0,5858 | 2,55 | 0,3421 |
| 0,20 | 0,8862 | 0,76 | 0,5827 | 2,60 | 0,3390 |
| 0,21 | 0,8772 | 0,77 | 0,5797 | 2,65 | 0,3360 |
| 0,22 | 0,8683 | 0,78 | 0,5767 | 2,70 | 0,3330 |
| 0,23 | 0,8596 | 0,79 | 0,5737 | 2,75 | 0,3302 |
| 0,24 | 0,8511 | 0,80 | 0,5708 | 2,80 | 0,3274 |
| 0,25 | 0,8427 | 0,81 | 0,5679 | 2,85 | 0,3247 |
| 0,26 | 0,8345 | 0,82 | 0,5651 | 2,90 | 0,3220 |
| 0,27 | 0,8264 | 0,83 | 0,5623 | 2,95 | 0,3194 |
| 0,28 | 0,8186 | 0,84 | 0,5596 | 3,00 | 0,3169 |
| 0,29 | 0,8108 | 0,85 | 0,5569 | 3,05 | 0,3144 |
| 0,30 | 0,8033 | 0,86 | 0,5542 | 3,10 | 0,3120 |
| 0,31 | 0,7959 | 0,87 | 0,5516 | 3,15 | 0,3097 |
| 0,32 | 0,7887 | 0,88 | 0,5490 | 3,20 | 0,3074 |
| 0,33 | 0,7816 | 0,89 | 0,5464 | 3,25 | 0,3051 |
| 0,34 | 0,7747 | 0,90 | 0,5439 | 3,30 | 0,3029 |
| 0,35 | 0,7680 | 0,91 | 0,5415 | 3,35 | 0,3008 |
| 0,36 | 0,7614 | 0,92 | 0,5390 | 3,40 | 0,2986 |
| 0,37 | 0,7550 | 0,93 | 0,5366 | 3,45 | 0,2966 |
| 0,38 | 0,7487 | 0,94 | 0,5342 | 3,50 | 0,2945 |
| 0,39 | 0,7425 | 0,95 | 0,5318 | 3,55 | 0,2926 |
| 0,40 | 0,7364 | 0,96 | 0,5295 | 3,60 | 0,2906 |
| 0,41 | 0,7305 | 0,97 | 0,5272 | 3,65 | 0,2887 |
| 0,42 | 0,7248 | 0,98 | 0,5249 | 3,70 | 0,2868 |
| 0,43 | 0,7191 | 0,99 | 0,5227 | 3,75 | 0,2850 |
| 0,44 | 0,7136 | 1,00 | 0,5205 | 3,80 | 0,2832 |
| 0,45 | 0,7082 | 1,05 | 0,5098 | 3,85 | 0,2814 |
| 0,46 | 0,7029 | 1,10 | 0,4998 | 3,90 | 0,2797 |
| 0,47 | 0,6977 | 1,15 | 0,4903 | 3,95 | 0,2780 |
| 0,48 | 0,6926 | 1,20 | 0,4814 | 4,00 | 0,2763 |
| 0,49 | 0,6876 | 1,25 | 0,4729 | 4,05 | 0,2747 |
| 0,50 | 0,6827 | 1,30 | 0,4649 | 4,10 | 0,2731 |
| 0,51 | 0,6779 | 1,35 | 0,4572 | 4,15 | 0,2715 |
| 0,52 | 0,6732 | 1,40 | 0,4499 | 4,20 | 0,2699 |
| 0,53 | 0,6686 | 1,45 | 0,4429 | 4,25 | 0,2684 |
| 0,54 | 0,6641 | 1,50 | 0,4363 | 4,30 | 0,2669 |
| 0,55 | 0,6596 | 1,55 | 0,4299 | 4,35 | 0,2654 |
| 0,56 | 0,6553 | 1,60 | 0,4238 | 4,40 | 0,2640 |
| 4,45 | 0,2625 | 4,65 | 0,2570 | 4,85 | 0,2519 |
| 4,50 | 0,2611 | 4,70 | 0,2557 | 4,90 | 0,2506 |
| 4,55 | 0,2597 | 4,75 | 0,2544 | 4,95 | 0,2494 |
| 4,60 | 0,2584 | 4,80 | 0,2531 | 5,00 | 0,2482 |

The general solution for the problem thus formulated, obtained by means of integral Hankel and Laplace transformations, may be written

$$\theta = \frac{T(r,\ x,\ \tau) - T_c}{T_0 - T_c} = 1 - \sum_{m=1}^{\infty} \frac{J_0\left(\mu_m \frac{r}{R}\right)}{\mu_m J_1(\mu_m)}$$

$$\times \left[ e^{-\mu_m k_1 X} \operatorname{erfc}\left(\frac{1}{2\sqrt{Fo_x}} - \mu_m k_1 X \sqrt{Fo_x}\right) + e^{\mu_m k_1 X}\right.$$

$$\left.\times \operatorname{erfc}\left(\frac{1}{2\sqrt{Fo_x}} + \mu_m k_1 X \sqrt{Fo_x}\right)\right] - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{J_0\left(\mu_m \frac{r}{R}\right)}{\mu_m J_1(\mu_m)}$$

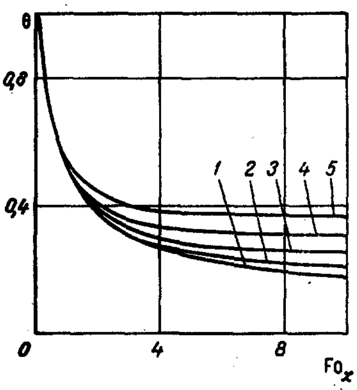


Fig. 3. Dependence of the relative excess temperature $\theta$ on the local $Fo_X$ number for the axis of a finite cylinder (r = 0) and various values of X and $k_1$: 1) semiinfinite body $k_1 = 0.5$, X = 0.15 with maximum relative error 0.4%; $k_1 = 0.5$, X = 0.2 (with 2.7%); $k_1 = 1.0$, X = 0.05 with 0.4%; $k_1 = 1.0$, X = 0.1 with 2.7%; $k_1 = 2.0$, X = 0.05 with 2.7%; 2) $k_1 = 1.0$, X = 0.15 and $k_1 = 0.5$, X = 0.3; 3) $k_1 = 1.0$, X = 0.2 and $k_1 = 2.0$, X = 0.1; 4) $k_1 = 1.0$, X = 0.25; 5) $k_1 = 2.0$, X = 0.15.

$$\times\left[e^{-\mu_m k_1(2n+X)}\,\mathrm{erfc}\left(\frac{\frac{2n}{X}+1}{2\sqrt{\mathrm{Fo}_x}}-\mu_m k_1 X\sqrt{\mathrm{Fo}_x}\right)\right.$$

$$+e^{\mu_m k_1(2n+X)}\,\mathrm{erfc}\left(\frac{\frac{2n}{X}+1}{2\sqrt{\mathrm{Fo}_x}}+\mu_m k_1 X\sqrt{\mathrm{Fo}_x}\right)$$

$$-e^{-\mu_m k_1(2n-X)}\,\mathrm{erfc}\left(\frac{\frac{2n}{X}-1}{2\sqrt{\mathrm{Fo}_x}}-\mu_m k_1 X\sqrt{\mathrm{Fo}_x}\right)$$

$$\left.-e^{\mu_m k_1(2n-X)}\,\mathrm{erfc}\left(\frac{\frac{2n}{X}-1}{2\sqrt{\mathrm{Fo}_x}}+\mu_m k_1 X\sqrt{\mathrm{Fo}_x}\right)\right], \tag{1}$$

$\mu_m$ are the roots of the equation $J_0(\mu) = 0$.

From Eq. (1) it is quite easy to derive a number of particular solutions. Putting $R \to \infty$, $k_1 \to 0$ we obtain a solution for an infinite plate with boundary conditions $T(0, \tau) = T_c$, $T(l, \tau) = T_0$, i.e.,

$$\theta = \frac{T(x,\tau)-T_c}{T_0-T_c} = \mathrm{erf}\,\frac{1}{2\sqrt{\mathrm{Fo}_x}}$$

$$-\sum_{n=1}^{\infty}\left(\mathrm{erfc}\,\frac{\frac{2n}{X}+1}{2\sqrt{\mathrm{Fo}_x}}-\mathrm{erfc}\,\frac{\frac{2n}{X}-1}{2\sqrt{\mathrm{Fo}_x}}\right). \tag{2}$$

As $l \to \infty$, $X \to 0$, Eq. (2) yields a solution for a semiinfinite body subject to the boundary conditions $T(0, \tau) = T_c$, $\partial T(+\infty, \tau)/\partial x = 0$

$$\theta_\infty = \mathrm{erf}\,\frac{1}{2\sqrt{\mathrm{Fo}_x}}. \tag{3}$$

Equations (2) and (3) agree completely with the corresponding solutions derived by A. V. Lykov [1].

There is thus a reasonable probability that, up to a certain number $Fo_X$, for fixed values of X (a parameter characterizing the relative height of the temperature sensor) and $k_1$ (a parameter characterizing the relationship between the linear dimensions of the cylinder), the propagation of heat in a finite cylinder will take place in the same way as in an infinite body.

Before proceeding to compare Eq. (1) with Eq. (3), it is desirable, as a first approximation to our assessment of the semiinfinite nature of the conditions, to analyze the propagation of heat in an infinite plate, and to find those values of the parameter X for which the actual temperature field lies within a prespecified range of the corresponding temperature field of an infinite body for any desired $Fo_X$ number.

Figure 1 shows the manner in which the relative error $\Delta\theta/\theta_\infty$ (%) varies with the parameter X for an infinite plate and various values of the local Fourier number $Fo_X$. From the family of these curves we may find those values of the height of the test sample $l$ (as yet without considering heat transfer at the lateral surface) corresponding to a specified x for which the "semiinfinite" condition will be satisfied to an acceptable accuracy up to a particular instant of time $\tau$ corresponding to a specified value of $Fo_X$.

In order to estimate the influence of heat transfer at the lateral surface of the finite cylinder on the development of the temperature field in the latter, and ultimately to determine the deviation of this field from that of a semiinfinite body, we must specify current coordinates of the finite cylinder. In accordance with Fig. 1, calculations were carried out for X = 0.1 and X = 0.2. The current coordinate r was taken as zero (the points r = 0 are most distant from the lateral surface of the cylinder). Clearly for any other values of $r \neq 0$ the effect of heat transfer at the lateral surface of the cylinder will enter more substantially.

Figure 2 gives the two-dimensional dependence of the relative error $\Delta\theta/\theta_\infty$ on the parameter $k_1$ (for two values of the parameter X) on the axis of the finite cylinder for various Fourier numbers $Fo_X$. Since Eq. (1) was obtained for boundary conditions of the first kind, it is clear that the $\Delta\theta/\theta_\infty = f(k_1)$ relationships presented characterize the most unfavorable experimental condition, in which the lateral surface of the cylinder and the $x = l$ exchange heat with a medium still existing at the original temperature, the exchange being governed by a heat-transfer coefficient of $\alpha = \infty$. For smaller values of $\alpha$ the difference between the temperature fields of the finite and semiinfinite cylinders will be less marked.

Figure 3 shows the deviations of the two-dimensional temperature fields from the temperature field of a semiinfinite body (curve 1); the final effect of the development of these two-dimensional temperature fields will be a steady thermal state with a nonzero steady component. Thus the heat transfer of a finite cylinder with specified boundary conditions may be regarded as a process involving steadily-acting heat sources and sinks.

Allowing for the foregoing assessments of the "semiinfinite" state of the system, Eq. (3) enables us to determine the thermal diffusivity of the material. For this purpose we may use Table 1, provided that we know the $T = f(\tau)$ relationship at a given point x from experiments with a model of a semiinfinite body, the specified boundary conditions being realized. Thus a specific number $Fo_X$ and a specific value of the relative excess temperature $\theta_\infty$ will correspond to a specific instant of time $\tau_e$ (let us call this the calculated time of the experiment). The thermal diffusivity is determined from the equation

$$a = \mathrm{Fo}_x \frac{x^2}{\tau_c}. \tag{4}$$

In conclusion, it should be mentioned that the maintenance of a steady temperature at the surface of the semiinfinite body is equivalent to the specification (on the same surface) of a continuously-varying thermal flux $q(\tau)$, the law of variation of which is easy to find if we know grad $T = f(\tau)$ at $x = 0$.

After differentiating Eq. (3) with respect to x, we obtain

$$q(\tau)|_{x=0} = \frac{T_0 - T_c}{\sqrt{\pi}} \cdot \frac{\lambda}{\sqrt{a\tau}}. \tag{5}$$

Thus all our conclusions as to the satisfaction of the "semiinfinite" condition based on the solution of (1)-(3) for a boundary condition of the first kind will also be valid for problems involving semiinfinite bodies with boundary conditions of the second kind.

## NOTATION

| | |
|---|---|
| $T(r, x, \tau)$ and $T_0$ | temperature at any point of a finite cylinder at any instant of time and initial temperature; |
| r and x | current coordinates of the finite cylinder; |
| $l$ and R | height and radius of the finite cylinder; |
| $J_0$ and $J_1$ | Bessel functions of the zeroth and first order and the first kind; |
| $\mu_m$ | roots of the Bessel function of the zeroth order and first kind; |
| $k_1 = l/R$ | parameter characterizing the relation between the linear dimensions of the finite cylinder; |

| | |
|---|---|
| $X = x/l$ | parameter characterizing the relative height of the temperature sensor; |
| $\tau$ | time; |
| $\lambda$ and $a$ | thermal conductivity and thermal diffusivity; |
| $Fo_X = a\tau/x^2$ | local Fourier number; |
| $\theta_\infty = (T(x, \tau) - T_C)/(T_0 - T_C)$ | relative excess temperature in a semiinfinite body; |
| $\Delta\theta = \lvert\theta - \theta_\infty\rvert$ | difference between the relative excess temperatures on the axis of a finite cylinder (and also an infinite plate) and a semiinfinite body at a specified point x; |
| $\operatorname{erf} z = 2/\sqrt{\pi}\int_0^z e^{-z^2}dz$ | Gauss error function; |
| $\operatorname{erfc} z = 1 - \operatorname{erf} z = 2/\sqrt{\pi}\int_z^\infty e^{-z^2}dz$; | |
| $q(\tau)$ | thermal flux density. |

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